Basic Framework for Games with Quantum-like Players

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Outline:

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4. Applied example
5. General framework
6. Solution concept

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Highlights

**Concept:**
- Agents are measurable systems.

**Framework:**
- Hilbert Space of Types.

**Departure from ‘classical’ models:**
- Preferences are realized in the elicitation process.
- End-nodes are no longer the carriers of utility.

**Main contribution:**
- Strategic manipulation of rivals’ preferences.
- Correlation between decision-situations.
**Behavioral pattern:**

“... in situations of violation of procedural invariance, observed preferences are not simply read off from some master list; they are actually constructed in the elicitation process.”

Kahneman and Tversky

*Choice, Values and Frames* (2000, p.504)
State of Preferences and Framing of the Problem

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State of Preferences and Framing of the Problem

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A model of a measurable agent includes:

- A set of states $S$,

- An outcome mapping, $\mu_M : S \to \Delta[O(M)]$, for every measurement $M \in \mathcal{M}$.

- A transition mapping, $\tau_{M,o} : S \to S$, for every measurement $M \in \mathcal{M}$ and any of its outcome $o \in O(M)$.

Danilov and Lambert-Mogiliansky
Type Indeterminacy: A model of the Kahneman-Tversky-man.

Quantum Type Indeterminacy in Dynamic Decision-Making: Self-Control through Identity Management.
Lambert-Mogiliansky and Busemeyer, Games, 3, 97-118 (2012).

An Empirical Test of Type-Indeterminacy in the Prisoner's Dilemma.
Hilbert Space of Types

(Basics)
Hilbert Space of Types

- Preference relations $\rightarrow$ Vector basis.
- Vector basis $\rightarrow$ Space of Types.
- Space of Types $+$ Set of Measurements
Hilbert Space of Types

Postulate (Superposition of Preferences)

Let \( \mathcal{T}_i \) be the Hilbert space of types spanned by the possible preferences of the player, \( \{ \theta_i^{(n)} \}_{n=1}^N \). Then, every type of the form

\[
t_i = \sum_{n=1}^{N} c_n \theta_i^{(n)}, \text{ with } c_n \in \mathbb{R}, \text{ and } \sum_{n=1}^{N} c_n^2 = 1,
\]

is also a unit length vector belonging to the same Hilbert space, \( t_i \in \mathcal{T}_i \). Hence, any linear combination of preferences (eigentypes) of a player is itself a proper type of the player.
Hilbert Space of Types

**Definition (Action as projector)**

Let \( a_i(d) \in A_i(d) \) be a particular action that can be chosen by player \( i \) at decision-situation \( d \), and let us assume that \( a_i(d) \) is the preferred action for a certain number \( M \leq N \) of eigentypes \( \{t_1, \ldots, t_M\} \in \Theta_i(d) \). Then, the matrix

\[
P_{a_i}(d) = \sum_{t_1, \ldots, t_M} t_m t_m^T
\]

(2)

is the projector associated to action \( a_i(d) \).\(^a\)

\(^a\)The row-vector \( t_m^T \) is the transposition of \( t_m \in \mathcal{T}_i \), and \( P_{a_i}(d) \) is a \( N \times N \) matrix.
Hilbert Space of Types

**Postulate (Actions affect Preferences)**

For player $i$ when facing a given decision-situation $d$, the chosen action $a_i(d) \in A_i(d)$ is the outcome of a measurement of her preferences. The type of player $i$ after making her decision in decision-situation $d$ is:

$$t'_i = \frac{P_{a_i(d)}t_i}{\|P_{a_i(d)}t_i\|}$$

(3)

where $t_i$ is the type before making the decision.
Applied example
Applied example: *Alice, Bob and the PhD Thesis*...
Applied example

The **space of types** is such that Player $A$ has a unique and trivial type, while $T_B = \text{Span}\{\theta_1, \theta_2\} = \text{Span}\{\tau_1, \tau_2\}$.

Decomposition for Player $A$ is trivial in the sense that there is no indeterminacy relevant to the game for this player.

Player $B$ presents two sets of eigentypes, associated to the measurements of $\theta$-preferences in nodes 2-3 and of $\tau$-preferences in nodes 8-11, with

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 
\end{pmatrix} = \begin{pmatrix}
\sqrt{0.4} & \sqrt{0.6} \\
\sqrt{0.6} & -\sqrt{0.4}
\end{pmatrix} \begin{pmatrix}
\tau_1 \\
\tau_2 
\end{pmatrix}
$$

(4)

giving the **relationship** between the **orthonormal basis**.
Applied example

**Initial state** of Player $B$ we have $t^0_B = (\sqrt{0.6}, \sqrt{0.4})\theta$, as given parameters. From (4), $t^0_B$ is given in coordinates of the $\tau$-basis of eigenpreferences by $t^0_B = (\sqrt{0.96}, \sqrt{0.04})\tau$.

Specification of the payoffs:

(i) The **payoffs** that both players will receive from their collaboration in the **Master thesis** are given in the following two matrices (Bob, Alice):

$$
\begin{array}{c|cc}
\theta_1 & S & I \\
\hline
R & (0; 0) & (-10; 20) \\
C & (-5; 0) & (-15; 70)
\end{array}
\quad \text{and} \quad
\begin{array}{c|cc}
\theta_2 & S & I \\
\hline
R & (0; 0) & (5; 20) \\
C & (-5; 0) & (10; 70)
\end{array}
$$

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(ii) For doing the PhD, Bob will get a fixed amount of utility $u$ due to the degree he earns and the scholarship he receives, regardless of who is his supervisor.

For Alice, who really wants to see her new idea developed, if Bob agrees ($A$) to work with her, she will receive $u_A(A, Inv) = 200$ or $u_A(R, Inv) = 0$ if he rejects and goes with another advisor.
Applied example

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Applied example

Solving for the second stage, the \( \{\tau_1, \tau_2\} \) are choice-making types:

- \( \tau_1 \) is open-minded and willing to work with Alice.
- \( \tau_2 \) will not trust her very specific idea and will refuse her as PhD advisor.

Solving for the first one:

- \( \theta_1 \)-type contribution to Bob’s personality makes him prefer a routine \( (R) \) solution to a standard task \( (S) \) and to the intricate one \( (I) \).
- \( \theta_2 \)-type contribution wills to give a routine \( (R) \) solution to a standard task \( (S) \) but a creative solution to an intricate problem \( (I) \).
Applied example

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Applied example

Utility levels $u_A[\sigma_1] = 0.96 \cdot 200 = 192$, and $u_B[\sigma_1] = u$ are trivial.

For the second path,

$$u_A[\sigma_2] = p_3(\theta_1) \left\{ u_A(R,I) + p_{10}(\tau_1)u_A(A,Inv) + p_{10}(\tau_2)u_A(R,Inv) \right\}$$

$$+ p_3(\theta_2) \left\{ u_A(C,I) + p_{11}(\tau_1)u_A(A,Inv) + p_{11}(\tau_2)u_A(R,Inv) \right\}$$

gives

$$u_A[\sigma_2] = 0.6(20+0.4\cdot200+0.6\cdot0)+0.4(70+0.6\cdot200+0.4\cdot0) = 136.$$
Applied example

and

\[
\begin{align*}
    u_B[\sigma_2] &= p_3(\theta_1) \left\{ u_B(\theta_1, R, I) + p_{10}(\tau_1)u_B(\tau_1, A, Inv) + p_{10}(\tau_2)u_B(\tau_2, R, Inv) \right\} \\
    &+ p_3(\theta_2) \left\{ u_B(\theta_2, C, I) + p_{11}(\tau_1)u_B(\tau_1, A, Inv) + p_{11}(\tau_2)u_B(\tau_1, R, Inv) \right\}
\end{align*}
\]

(7)

gives

\[
u_B[\sigma_2] = 0.6(-10 + 0.4 \cdot u + 0.6 \cdot u) + 0.4(10 + 0.6 \cdot u + 0.4 \cdot u) = u - 2.
\]
General framework
General framework

**Definition (TI player)**

A Type Indeterminate player $i$ is a decision-maker facing a set of decision-situations $D_i$, for which player $i$’s preferences are actualized when the decision is made, interpreted as the outcome of a measurement process.
Elements of a TI player:

- The set $D_i$ of decision-situations in which player $i$ has to take an action.
- The family $\Theta_i = \{\Theta_i(d)\}_{d \in D_i}$ of the sets $\Theta_i(d)$ of eigentypes of player $i$ giving the orthonormal basis of eigenpreferences of the player for each decision-situation $d \in D_i$.
- The initial type of the player, $t_i^0 \in T_i$, given as an element of the Hilbert space of types $i$ spanned by the basis of the decision-situations corresponding to player $i$, $\Theta_i$. 

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## General framework

### Definition (Maximal information)

A TI game is of maximal information if:

- Every decision-situation of the game is identified and common knowledge.
- Every orthonormal basis \( \Theta_i(d) \in \Theta_i \) associated to each decision-situation \( d \in D_i \) is common knowledge, as well as the relation among them, \( B_i^{(d'; d)} \) for every pair \( d, d' \in D_i \).
- The initial type \( t_i^0 \) of every player is known and common knowledge.
Solution concept
Solution concept

**Elements:**
- Paths of Projection.
- Evolution of Preferences.
- Cashing-on-the-go.
Solution concept

Computation:

- First, we define a TINE as the profile of the (pure strategy) **best-reply of all the eigentypes** of all players, a complete algorithm for the players’ action in the game.

- Second, we compute the overall **utility of the paths**. We propose *cashing-on-the-go* for solving TI games as an Assumption.

- Third, we compute the profile of **outcoming types** of the players which, in general, are different from the initial types. The play alters preferences in accordance with the Postulate of Projection previously discussed.
Type Indeterminate Nash Equilibrium (Formalities)
Solution concept - I. Definition of TINE:

**Definition (TINE)**

A particular profile of strategies \((s_i^*; s_{-i}^*)\) constitutes a Type Indeterminate Nash Equilibrium of a TI game when they are the collection of all the eigentypes of every player best-responding to every eigentype of the other players, so that

\[
U_i[\theta_i^{(n)}, a_i^*(d), a_{-i}^*(d)] \geq U_i[\theta_i^{(n)}, a'_i(d), a_{-i}^*(d)] \quad \forall a'_i(d) \tag{8}
\]

holds for every eigentype \(\theta_i^{(n)} \in \Theta_i(d)\) of every player \(i \in I\), at every decision-situation \(d \in D_i\).
Solution concept

The outcome of the TINE includes:
- the overall utility level of every TI player,
- and the final state of preferences of every TI player.
Let \( M \leq N \) be the preference relations (eigentypes) supporting a particular action as their preferred one in the strategic interaction.

Then the outcoming type of the player after the interaction is given by:

\[
t'_i = \sum_{m=1}^{M} c'_m \theta^{(m)}_i, \quad \text{with } c'_m \in \mathbb{R}, \quad \text{and } \sum_{m=1}^{M} c'^2_m = 1.
\]  \hspace{1cm} (9)
Solution concept - II. Utility of the outcome:

Definition (Cashed utility of a preferred action)

Let player $i$ face decision-situation $d$ with preferences given by a type $t_i \in T_i$, and let $a^*_i(d)$ be the chosen actions by the other players. Then, the utility *cashed* by player $i$ when taking action $a^*_i(d)$ as in Definition ?? is the weighted utility of the $M$ eigenpreferences where the weights are given by the coefficients of superposition:

$$u_i[a^*_i(d), a^*_{-i}(d); t'_i] = \sum_{m=1}^{M} c^2_m u_i[\theta_i^{(m)}, a^*_i(d), a^*_{-i}(d)].$$  (10)
Solution concept

Assumption

We compute the overall utility of the players involved in a TI game as the addition of the local utilities cashed by the type of each of the players in the decision-situations along the path of play.
Definition (Total utility of a path)

Let $\sigma_l \in \Sigma$ be a path $\sigma_l = \{d_1, \ldots, d_l, \ldots, d_L\}$, induced by strategy profile $(s_i^*; s_{-i}^*)$. Then, the total utility for player $i$ in the play following path $\sigma_l$ is given by

$$u_i[\sigma_l] = \sum_{d_l \in \sigma_l} u_i[a_i^*(d_l), a_{-i}^*(d_l)] ,$$  \hspace{1cm} (11)

with the utility of the preferred actions given by $u[\cdot]$. 
Solution Concept - III. Outcoming preferences:

**Definition**

Given a path of play $\sigma_l$, and the state-vector of initial preferences of a TI player $i$ denoted by $t^0_i \in T_i$, the outcoming preferences of the TI player $i$ are computed by the subsequent projections as a consequence of the actions taken along the path of play:

$$t^{(\text{end})}_i = \prod_{d_l \in \sigma_l} P_{a^*_i(d_l)} t^0_i$$

for every player $i \in I$. (12)

Composition of operators: $P_{a^*_i(d_L)} \cdots P_{a^*_i(d_2)} P_{a^*_i(d_1)} t^0_i$.
Some Open Questions:

- How to build $U$: constant discount rate, hyperbolic discounting, myopia...
- Clarify decision-utility vs experienced-utility.
- Insights from psychology and behavioral analysis to build $B$.
- Extension to incomplete information: density matrix.
Final discussion

- Type Indeterminacy introduces endogenous preferences.
- Strategic interaction is not only useful to predict movements, but also to influence and manipulate them by means of your own previous choices.